Wigner function as representation for tunneled electron

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I. INTRODUCTION

Quantum pumps [2, 7] recently have been proposed for optics-like experiments as on-demand electron source, where the properties of emitted electrons are still subject for study.

Typical optics-like experiment consists of on-demand electron source [6, 10, 12] quantum Hall edge channel and beam-splitter and uses March-Zeinder interferometer geometry [11] to obtain information about emitted electron's quantum state. Since electrons on quantum Hall edge channel behaves as free particles [3] then it is reasonable to assume that they remain coherent by the time of detection. Therefore it is possible to study the whole system from two parts where one is electron source with quantum Hall edge channel.

Theoretical framework consists from properties of low temperature free electron gas and its excitations. These excitation are the ones which quantum Hall edge channel delivers to the observer (read spectrometer or beam-splitter). On quantum Hall edge channel the excited electron (further electron) is characterized with its position x and momentum p, where both are tied with uncertainty principle $\Delta x \Delta p \leq \hbar/2$, therefore electron must be represented either with quantum state vector or with Wigner function [4]. This electron state however is made at the time of emission giving us opportunity to modify it by modifying emission protocol where on this summary quantum pump have been considered.

II. THE TUNNELING PROBLEM FROM QUANTUM PERSPECTIVE

Since electron before and after tunneling has the same energy, then knowing how energy changes on quantum dot with additionally with time dependence of tunneling rate gives all what is needed to characterize excitation on Hall edge channel.

The whole system in this case can be modeled from two parts - quantum dot and lead - where interaction is modeled as mixing of quantum states. We are assuming that this interaction does not depend on energy of quantum dot or in other words mixing between dot and lead states are equal. Therefore this case is modeled with Hamiltonian:

$$\boxed{\hat{H} = \epsilon_d(t) \left|d\right\rangle \left\langle d\right| + \sum_k \epsilon_k \left|k\right\rangle \left\langle k\right| + + \sum_k \left[V^*(t) \left|d\right\rangle \left\langle k\right| + V(t) \left|k\right\rangle \left\langle d\right|\right]} \tag{1}$$

where $|d\rangle$ is state of the dot, $|k\rangle$ states of the lead, $V(t)$ is interaction term and summation is along all possible wave-vectors in the lead. The interaction term $V(t)$ is given according to Fermi golden rule $\Gamma_k = 2\pi\delta(\epsilon_k - \epsilon_d)|V|^2$ as:

$$V(t) = \sqrt{\frac{\Gamma(t)}{2\pi\rho}} \tag{2}$$

where ρ is energetic density of states but $\Gamma(t)$ is defined according to kinetic equation for probability for electron to be in the dot P_d as:

$$\frac{dP_d}{dt} = -\frac{\Gamma(t)}{\hbar} P_d(t) \tag{3}$$

The Hamiltonian suggests that quantum state vector can be represented as:

$$\left|\Psi(t)\right\rangle = c_d(t) \left|d\right\rangle + \sum_k c_k(t) \left|k\right\rangle \tag{4}$$

where c_d, c_k are amplitudes of the states, Since emission process is of our interest then initially we are assuming that on quantum dot fully sits an electron or:

$$c_d(t_0) = 1 \tag{5}$$

$$c_k(t_0) = 0,\ k = \textit{all possible wavevectors} \tag{6}$$

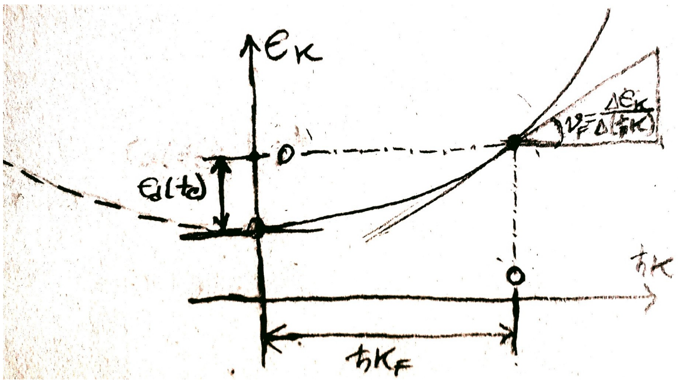

FIG. 1: Schematics of dispersion and its linearisation[a].

[a] Seems wrong when I am reffering $\epsilon_d(t_e)$ to energy interval. Therefore image will be improved.

With this initial condition and time dependent Schroedinger equation one obtains the solution for lead state amplitudes [1, 9][1,2]:

$$\boxed{c_k(t) = \frac{-i}{\hbar\sqrt{2\pi\rho}} \int_{t_0}^{t} \sqrt{\Gamma(t_1)} \exp\left[\frac{-i}{\hbar} \int_{t_0}^{t_1} \epsilon_d(\tilde{t}) - \frac{i}{2}\Gamma(\tilde{t}) d\tilde{t}\right] e^{-i\epsilon_k(t-t_1)/\hbar} dt_1} \tag{7}$$

A. Rewriting with wave-vectors

When electron tunnels out of the dot it becomes part of Hall edge channel which is characterized with dispersion shown in Figure 1. According to it the electron energy in the lead can be given in terms of wave-vectors, where we for simplicity consider only linear part around some characteristic quantum dot energy $\epsilon_d(t_e)$. Referring energy and electron momentum around that point we can do linearization:

$$\epsilon_k = \left.\frac{d\epsilon_k}{d(\hbar k)}\right|_{\epsilon_d(t_e)} \hbar k = v_F \hbar k \tag{8}$$

Since we can write momentum operator as $\hat{p} = \hat{H}/v_F$ then energy state $|k\rangle$ is also momentum state for which can calculate amplitude after putting linearization in (7):

$$c_k(t) = \frac{-i}{\hbar\sqrt{2\pi\rho}} \int_{t_0}^{t} \sqrt{\Gamma(t_1)} \exp\left[-\frac{i}{\hbar} \int_{t_0}^{t_1} \epsilon_d(\tilde{t}) - \frac{i}{2}\Gamma(\tilde{t}) d\tilde{t}\right] e^{+iv_F k(t_1-t)} dt_1 \tag{9}$$

B. Approximation for full emission

Let's set initial condition $t_0 \to -\infty$ but time t we are considering large enough to be certain that electron on Hall edge channel has been fully emitted. From mathematical perspective I am assuming that integrated function in (9) is localized for t_1 and therefore can be approximated as:

$$c_k(t) \overset{X(t)\gg 1}{=} \frac{-i}{\hbar\sqrt{2\pi\rho}} e^{-iv_F kt} \int \sqrt{\Gamma(t_1)} \exp\left[-\frac{i}{\hbar} \int_{t_0}^{t_1} \epsilon_d(\tilde{t}) - \frac{i}{2}\Gamma(\tilde{t}) d\tilde{t}\right] e^{+iv_F k t_1} dt_1 \tag{10}$$

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[1] However both are considered with $t_0 = 0$ instead of arbitrary initial time. In my Bachelour thesis I showed how to shift the solution, but it is almost as long as derivation itself making me to consider of including derivation. The only misunderstunding to me is in [1] equation [$A3$]!

[2] We can also express it as

$$c_k(t) = -\frac{i}{\sqrt{\hbar\rho}} \int_{t_0}^{t} \sqrt{p_t(t_1)} \exp\left[-\frac{i}{\hbar} \int_{t_0}^{t_1} \epsilon_d(\tilde{t}) d\tilde{t} - \frac{i}{\hbar}\epsilon_k(t-t_1)\right] dt_1$$

where $p_t(t)$ is time probability density distribution from (3) as $p_t(t) = \frac{dP_d}{dt}$

For what follows the solution is also rewritten as[3]:

$$c(k,t) = \frac{-i}{\hbar\sqrt{2\pi\rho}} \int B(t_1) e^{+iv_F k(t_1-t)} dt_1 \tag{11}$$

$$B(t) = \sqrt{\Gamma(t_1)} \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t_1} \epsilon_d(\tilde{t}) - \frac{1}{2}X(t)\right] \tag{12}$$

$$X(t) = \frac{1}{\hbar}\int_{-\infty}^{t} \Gamma(\tilde{t}) d\tilde{t} \tag{13}$$

where I have emphasized that the solution of $c_k(t)$ is continuous function for wave-vectors k.

III. PROPERTIES AND DEFINITION OF WIGNER FUNCTION

The function $W(x,p)$ which gives expectation value on phase space $\tilde{A}(x,p)$ in the following way:

$$\langle \hat{A} \rangle = \int W(x,p)\tilde{A}(x,p) dx dp \tag{14}$$

is called Wigner function. The operator $\tilde{A}(x,p)$ however is connected with quantum mechanical analog $\hat{A}$ with Weyl transform:

$$\tilde{A}(x,p) = \int e^{-ipy/\hbar} \langle x + y/2 | \hat{A} | x - y/2 \rangle dy \tag{15}$$

Defined in this way the following property is valid [4]:

$$\mathrm{Tr}\, \hat{A}\hat{B} = \frac{1}{h}\int \tilde{A}(x,p)\tilde{B}(x,p) dx dp \tag{16}$$

Since expectation value for operator is also given with density operator $\hat{\rho}$ as $\langle \hat{A} \rangle = \mathrm{Tr}\, \hat{\rho}\hat{A}$ then using property above one obtains the formula for Wigner function from:

$$\int W(x,p)\tilde{A}(x,p) dx dp \overset{(14)}{=} \mathrm{Tr}\, \hat{\rho}\hat{A} \overset{(16)}{=} \frac{1}{h}\int \tilde{\rho}(x,p)\tilde{A}(x,p) dx dp \tag{17}$$

and from which follows:

$$\boxed{W(x,p) = \frac{1}{h}\tilde{\rho}(x,p) = \frac{1}{h}\int e^{-ipy/\hbar} \langle x + y/2 | \psi \rangle \langle \psi | x - y/2 \rangle dy} \tag{18}$$

Two properties follows immediately from definition [4]:

$$p_x(x) = \int W(x,p) dp \tag{19}$$

$$p_p(p) = \int W(x,p) dx \tag{20}$$

where p_x is position but p_p momentum probability density. For Wigner function there is also property for momentum shift. For example if we consider wave function $e^{-ixk_F}\psi(x)$ then the new Wigner function according to old one becomes $W(x, \hbar k + \hbar k_F)$. It gives us possibility to use shifted wave function directly to plot Wigner function. For example if we consider free particle wave function in the box of size $-L/2 \le x \le +L/2$ then its shifted wave function becomes:

$$\psi_k(x) = e^{-ik_F x}\frac{1}{\sqrt{L}} e^{+i(k+k_F)x} = \frac{1}{\sqrt{L}} e^{+ikx} \tag{21}$$

where we only consider electrons which moves away from quantum dot in the direction of x-axis (accounted with + sign in exponent).

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[3] Might not be the right place. Also since in Wigner function it will appear multiplied with complex conjugate then for simplicity of derivations one can set $t_0 = 0$ for integral of $\epsilon_d(t_e)$

IV. WIGNER FUNCTION FROM THE SOLUTION OF SCHROEDINGER PICTURE

The energy states are also momentum states for free particles and by transformation can be obtained from position basis. By putting quantum dot at $x = 0$ with size of system L we can define identity operator either in momentum and position states:

$$\hat{1} = \sum_k |k\rangle \langle k| = \int_{-L/2}^{+L/2} |x\rangle \langle x| \, dx \tag{22}$$

where orthonormality of states as $\langle k|k'\rangle = \delta_{kk'}$ and $\langle x|x'\rangle = \delta(x - x')$ follows. Now we are considering by $\hbar k_F$ shifted wave function of momentum state $|k\rangle$ of free particle:

$$\langle x|k\rangle = \psi_k(x) = \frac{1}{\sqrt{L}} e^{+ikx} \tag{23}$$

Applying identity operator on position and momentum states and using property above transformation formulas follows:

$$|x\rangle = \hat{1} |x\rangle = \sum_k |k\rangle \otimes \langle k| \, |x\rangle = \sum_k \langle k|x\rangle \, |k\rangle = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \, |k\rangle \tag{24}$$

$$|k\rangle = \hat{1} |k\rangle = \int_{-L/2}^{+L/2} |x\rangle \langle x| \, dx \, |k\rangle = \int_{-L/2}^{+L/2} \langle x|k\rangle \, |x\rangle \, dx = \frac{1}{\sqrt{L}} \int_{-L/2}^{+L/2} e^{+ikx} \, |x\rangle \, dx \tag{25}$$

For checking one puts them back in the identity operator and uses that delta functions can be also expressed as:

$$\delta_{kk'} = \frac{1}{L} \int_{-L/2}^{+L/2} e^{-i(k-k')x} dx \tag{26}$$

$$\delta(x - x') = \frac{1}{L} \sum_k e^{-ik(x-x')} \tag{27}$$

which are valid for $k = \frac{2\pi}{L} \cdot integer$.

A. Wide band approximation

For practical uses one usually approximates summation along momentum states $|k\rangle$ with integration:

$$\sum_k f(k) = \frac{1}{\Delta k} \sum_k f(k) \Delta k \approx \frac{1}{\Delta k} \int f(k) dk \tag{28}$$

where Δk is wavenumber spacing in the system. Let's assume that our function is Kronecker delta $\delta_{qq'}$ then putting in the last equation it's continuous analog (26) we obtain:

$$\sum_q \delta_{qq'} \approx \int \left(\frac{1}{\Delta q L} \int_{-L/2}^{+L/2} e^{-i(q-q')x} dx \right) dq = \frac{2\pi}{\Delta q L} \int_{-L/2}^{+L/2} e^{+iq'x} \delta(x) dx = \frac{2\pi}{\Delta q L} \tag{29}$$

and knowing that left side equals to 1 for any q' then one determines spacing between wave vectors Δk in terms of geometric system size L:

$$\Delta k = \frac{2\pi}{L} \tag{30}$$

It immediately shows that no approximations is made on evaluating the sum with integral when length of the system becomes infinite long or $L \to +\infty$.

It is also possible to estimate energetic density of states or $\rho = 1/\Delta \epsilon_k$ in terms of L. By differencing (8) to $\Delta \epsilon_k = v_F \hbar \Delta k$ and using (30) we can express density of energy states as:

$$\rho = \frac{1}{\Delta \epsilon_k} = \frac{1}{v_F \hbar \Delta k} = \frac{L}{v_F h} \tag{31}$$

B. Wigner function from state amplitudes

The wave function from state vector (4) is obtainable as scalar product with position state $|x\rangle$:

$$\boxed{\psi(x) = \langle x|\Psi(t)\rangle = \sum_k c_k(t)\langle x|k\rangle = \frac{1}{\sqrt{L}}\sum_k c_k(t)e^{+ikx}} \tag{32}$$

which numerically can be computed as matrix multiplication with DFT matrix. Considering the limit $L \to \infty$ and using (28) we can express the wave function as integral:

$$\psi(x) = \frac{1}{\Delta k\sqrt{L}}\int c_k(t)e^{+ikx}dk = \frac{\sqrt{L}}{2\pi}\int c_k(t)e^{+ikx}dk \tag{33}$$

Putting it in definition of Wigner function (18) one expresses it from momentum states:

$$\begin{aligned}
W^t(x,\hbar k) &= \frac{1}{h}\int e^{-iky}\langle x+y/2|\psi\rangle\langle\psi|x-y/2\rangle dy \\
&= \frac{L}{4\pi^2 h}\int e^{-iky}e^{+iq(x+y/2)}c(q,t)e^{-iq'(x-y/2)}c^*(q',t)dydq'dq \\
&= \frac{L}{4\pi^2 h}\int c(q,t)c^*(q',t)e^{-iy(k-\frac{q+q'}{2})}e^{+ix(q-q')}dydq'dq \\
&= \frac{L}{4\pi^2 h}\int c(\xi_1+\xi_2/2,t)c^*(\xi_1-\xi_2/2,t)e^{-iy(k-\xi_1)}e^{+ix\xi_2}dyd\xi_1 d\xi_2 \\
&= \frac{L}{2\pi h}\int c(\xi_1+\xi_2/2,t)c^*(\xi_1-\xi_2/2,t)\delta(k-\xi_1)e^{+ix\xi_2}d\xi_1 d\xi_2 \\
&= \frac{L}{2\pi h}\int c(k+\xi_2/2,t)c^*(k-\xi_2/2,t)e^{+ix\xi_2}d\xi_2
\end{aligned} \tag{34}$$

V. TIME DEPENDANT WIGNER FUNCTION

In this section we are going to look how the Wigner function looks when the solution (9)with $t_0 \to -\infty$ have been put in previous result. For simplicity one works with non-dimensionless quantities (since v_F is not precisely known) to which we can transform Wigner function as distribution in form:

$$W^t(x,\hbar k) \overset{def}{=} \frac{1}{\hbar}\bar{W}^{t/\tau}\left(\frac{x}{v_F\tau}, v_F\tau k\right) \tag{35}$$

where I have introduced τ as free parameter to characterize the scale of time which will be convenient when emission protocol will be substituted. From chosen transformation above the non-dimensionality follows:

$$x \to v_F\tau x \tag{36}$$

$$t \to \tau t \tag{37}$$

$$k \to k/v_F\tau \tag{38}$$

with which we can express Wigner function above:

$$\bar{W}^t(x,k) = \hbar W^{\tau t}(v_F\tau x, \frac{\hbar}{v_F\tau}k) \tag{39}$$

Now using (34) the transformed Wigner function is given as:

$$\bar{W}^t(x,k) = \frac{\rho\hbar}{2\pi\tau}\int c(\frac{k+\xi_2/2}{v_F\tau},\tau t)c^*(\frac{k-\xi_2/2}{v_F\tau},\tau t)e^{+ik\xi_2}d\xi_2 \tag{40}$$

for convenience we need to express $c(k,t)$ from dimensionless quantities:

$$c\left(\frac{k}{v_F\tau},\tau t\right) = -i\sqrt{\frac{\tau}{2\pi\rho\hbar}}e^{-ikt}\int_{-\infty}^{t}\sqrt{\frac{\tau}{\hbar}\Gamma(t)}\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t_1}\epsilon_d(\tilde{t})d\tilde{t} - \frac{1}{2\hbar}\int_{-\infty}^{t_1}\Gamma(\tilde{t})d\tilde{t}\right]e^{+ikt_1}dt_1 \tag{41}$$

Putting last two expressions together the Wigner function is expressed as:

$$\bar{W}^t(x,k) = \frac{1}{4\pi^2}\int d\xi_2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2$$
$$\times\, e^{-i\xi_2\left(t-x-\frac{t_1+t_2}{2}\right)}\sqrt{\frac{\tau}{\hbar}\Gamma(t_1)}\sqrt{\frac{\tau}{\hbar}\Gamma(t_2)}\exp\left[-\frac{i}{\hbar}\int_{t_2}^{t_1}\epsilon_d(\tilde{t})d\tilde{t} - \frac{1}{2\hbar}\int_{\infty}^{t_1}\Gamma(\tilde{t})d\tilde{t} - \frac{1}{2\hbar}\int_{\infty}^{t_2}\Gamma(\tilde{t})d\tilde{t}\right]e^{-ik(t_2-t_1)} \quad (42)$$

For practical computation of the Wigner function one separates the evaluation in following way[4]:

$$\bar{\epsilon}_d(t) = \frac{\tau}{\hbar}\epsilon_d(\tau t) \quad (43)$$

$$\bar{\Gamma}(t) = \frac{\tau}{\hbar}\Gamma(\tau t) \quad (44)$$

$$\gamma(t) = \sqrt{\bar{\Gamma}(t)}\exp\left[-i\int_0^t \bar{\epsilon}_d(\tilde{t})d\tilde{t} - \frac{1}{2}\int_{-\infty}^{t}\bar{\Gamma}(\tilde{t})d\tilde{t}\right] \quad (45)$$

$$\bar{\beta}(k,t) = \int_{-\infty}^{t}\gamma(t_1)e^{+ikt_1}dt_1 \quad (46)$$

$$\bar{W}^t(x,k) = \frac{1}{4\pi^2}\int e^{-i\xi_2(t-x)}\bar{\beta}(k+\xi_2/2,t)\bar{\beta}^*(k-\xi_2/2,t)d\xi_2 \quad (47)$$

A. Momentum probability distribution

Using the property (20) it is possible from Wigner function obtain the momentum distribution which in introduced dimensionless units is:

$$p_k(k) = \int \bar{W}(x,k)dx = \frac{1}{2\pi}|\bar{\beta}(k,t)|^2 = \frac{1}{2\pi}\left|\int_{-\infty}^{t}\sqrt{\bar{\Gamma}(t_1)}\exp\left[-i\int_0^{t_1}\bar{\epsilon}_d(\tilde{t})d\tilde{t} - \frac{1}{2}\int_{-\infty}^{t_1}\bar{\Gamma}(\tilde{t})d\tilde{t}\right]dt_1\right|^2 \quad (48)$$

B. Position probability distribution

Similarly using property (19) the position probability distribution in dimensionless units is obtained:

$$\begin{aligned}
p_x^t(x) &= \int \bar{W}^t(x,k)dk \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2\int d\xi_2\int dk e^{-ik(t_2-t_1)}e^{-i\xi_2(t-x-\frac{t_1+t_2}{2})}\gamma(t_1)\gamma^*(t_2) \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{t}d\zeta_2\int_{-2(t-\zeta_2)}^{+2t(t-\zeta_2)}d\zeta_1\int d\xi_2\int dk e^{-ik\zeta_1}e^{-i\xi_2(t-x-\zeta_2)}\gamma(\zeta_2-\zeta_1/2)\gamma^*(\zeta_2+\zeta_1/2) \\
&= \frac{1}{2\pi}\int_{-\infty}^{t}d\zeta_2\int_{-2(t-\zeta_2)}^{+2(t-\zeta_2)}d\zeta_1\int d\xi_2\delta(\zeta_1)e^{-i\xi_2(t-x-\zeta_2)}\gamma(\zeta_2-\zeta_1/2)\gamma^*(\zeta_2-\zeta_1/2) \\
&= \int_{-\infty}^{t}d\zeta_2\delta(t-x-\zeta_2)\gamma(\zeta_2)\gamma^*(\zeta_2) \\
&= \Theta(x)\gamma(t-x)\gamma^*(t-x)
\end{aligned} \quad (49)$$

Therefore the position probability distribution is:

$$p_x^t(x) = \Theta(x)\bar{\Gamma}(t-x)\exp\left[-\int_{-\infty}^{t-x}\bar{\Gamma}(\tilde{t})d\tilde{t}\right] \quad (50)$$

[4] The 0 is arbitrary choice for $\gamma(t)$ due to multiplicaction with complex conjugate in the expression of Wigner functiom.

We can check that it is normalized, by comparing cumulative probability for electron to be on the lead:

$$P_{lead}(t) = \int p_x^t(x)dx = \int_0^{+\infty} \bar{\Gamma}(x-t)\exp\left[-\int_{-\infty}^{x-t}\bar{\Gamma}(\tilde{t})d\tilde{t}\right]dx = \int_{-\infty}^{t}\bar{\Gamma}(t')\exp\left[-\int_{-\infty}^{t'}\bar{\Gamma}(\tilde{t})d\tilde{t}\right]dt' \tag{51}$$

From the form of position distribution one can see that for some region it is invariant for Galilean transformation. Also the sharpness with Heaviside Theta function is present which comes from Wide band approximations:

- Wave-vectors span $-\infty < k < +\infty$ therefore in the third line of (49) Dirac delta can be produced
- Interaction term $V(t)$ in (1) is independent of energy in the dot
- The density of states in the lead ρ is independent of state energy

VI. ASYMPTOTICS OF WIGNER FUNCTION AS $t \to \infty$

In the limit of large t the amplitudes for state vectors become periodic (10) which makes the wave function (32) invariant to Galilean transformation and therefore also the Wigner function. Since the Galilean invariance Wigner function at one instant gives all information need to characterize emitted electron.

Let's assume that at position x_O there are observer. Then accounting the time which is needed for electron to came from quantum dot to the observer as x_O/v_F and making use of $p = \epsilon_k/v_F$ one can define Wigner function on time and energy axes in a way:

$$W_b(t,\epsilon_k) \overset{def}{=} W^{x_O/v_F+t}(x_O, \epsilon_k/v_F) \tag{52}$$

Since Wigner function asymptotically becomes invariant against Galilean transformation then the right side of equation above becomes invariant against position of observer and we can also refer it to one snapshot:

$$W_b(t,\epsilon_k) = \lim_{x_O\to+\infty} W^{x_O/v_F}(x_O - v_F t, \epsilon_k/v_F) \tag{53}$$

which allows us to calculate it.

A. Calculation of it in the asymptotic case

Putting our observer far away of quantum dot or $x_O \to \infty$ we can calculate asymptotics of Wigner function by using (10) and (53):

$$\begin{aligned}
W_b(t,\epsilon_k) &= \lim_{x_O\to+\infty} W^{x_O/v_F}(x_O - v_F t, \epsilon_k/v_F)\\
&= \frac{L}{2\pi h}\int c(\epsilon_k/v_F\hbar + \xi_2/2, x_O/v_F)c^*(\epsilon_k/v_F\hbar - \xi_2/2, x_O/v_F)e^{+i(x_O - v_F t)\xi_2}d\xi_2\\
&= \frac{\rho h v_F}{2\pi h}\frac{1}{2\pi\hbar^2\rho}\int B(t_1)e^{+iv_F(\frac{\epsilon_k}{v_F\hbar}+\frac{\xi_2}{2})(t_1-\frac{x_O}{v_F})}B^*(t_2)e^{-iv_F(\frac{\epsilon_k}{v_F\hbar}+\frac{\xi_2}{2})(t_2-\frac{x_O}{v_F})}dt_1 dt_2 d\xi_2\\
&\overset{\xi_2\to\xi_2/v_F}{=} \frac{1}{h^2}\int B(t_1)B^*(t_2)e^{-i\epsilon_k(t_2-t_1)/\hbar}e^{-i\xi_2(t-\frac{\xi_1+\xi_2}{2})}dt_1 dt_2 d\xi_2\\
&= \frac{1}{h^2}\int B(\zeta_1 - \zeta_2/2)B^*(\zeta_1+\zeta_2/2)e^{-i\epsilon_k\zeta_2/\hbar}e^{-i\xi_2(t-\zeta_1)}d\zeta_1 d\zeta_2 d\xi_2\\
&= \frac{2\pi}{h^2}\int B(\zeta_1-\zeta_2/2)B^*(\zeta_1+\zeta_2/2)e^{-i\epsilon_k\zeta_2/\hbar}\delta(t-\zeta_1)d\zeta_1 d\zeta_2\\
&= \frac{1}{2\pi\hbar^2}\int B(t-\zeta_2/2)B^*(t+\zeta_2/2)e^{-i\epsilon_k\zeta_2/\hbar}d\zeta_2
\end{aligned} \tag{54}$$

And putting back $B(t)$ from (13) we obtain Wigner function expressed from emission protocol:

$$\boxed{\begin{aligned}
W_b(t,\epsilon_k) = \frac{1}{2\pi\hbar^2}&\int e^{-i\epsilon_k T/\hbar}\sqrt{\Gamma(t-T/2)\Gamma(t+T/2)}\\
&\times\exp\left[-\frac{1}{2\hbar}\int_{-\infty}^{t-T/2}\Gamma(\tilde{t})d\tilde{t} - \frac{1}{2\hbar}\int_{-\infty}^{t+T/2}\Gamma(\tilde{t})d\tilde{t} + \frac{i}{\hbar}\int_{t-T/2}^{t+T/2}\epsilon_d(\tilde{t})d\tilde{t}\right]dT
\end{aligned}} \tag{55}$$

where one sees that integral basically is Fourier transform which helps with numerical approximations.

B. Probability distributions

From the expression (55) one can also obtain time distribution at which electron enters in the observer[5]:

$$p_t(t) = \int W_b(t,\epsilon_k)d\epsilon_k = \frac{1}{\hbar}|B(t)|^2 = \frac{\Gamma(t)}{\hbar}\exp\left[-\frac{1}{\hbar}\int_{-\infty}^{t}\Gamma(\tilde{t})d\tilde{t}\right] \tag{56}$$

which is also the distribution which one obtains after solving kinetic equation (3), therefore the Wigner function also tels us probability when it has tunneled from the dot. Also energy distribution by applying (20) and using (8) can be obtained similarly:

$$p_\epsilon(\epsilon_k) = \int W_b(t,\epsilon_k)dt \tag{57}$$

VII. SPECIAL CASES

No clue where to put it[6]:

$$\Gamma = \Gamma_0 e^{-(E_b-\epsilon_d)/\Delta_b} \tag{58}$$

A. Instantaneous rise of tunneling barrier

First case we are going to consider instantaneous rise of tunneling barrier [5] which as emission protocol can be given:

$$\Gamma(t) = \Gamma_0\Theta(t) \tag{59}$$

$$\epsilon_d(t) = \epsilon_d^-(t)\Theta(-t) + \epsilon_d^+\Theta(t) \tag{60}$$

Since electron does not tunnel for times $t<0$ then $\epsilon_d^-(t)$ does not contribute to the quantum state in the lead. Also we will consider energy reference frame at which $\epsilon_d^+ = 0$ therefore considered emission protocol is simplified to:

$$\Gamma(t) = \Gamma_0\Theta(t) \tag{61}$$

$$\epsilon_d(t) = 0 \tag{62}$$

Before we are putting it to the Wigner function we will express $B(t)$ from (13)

$$B(t) = \Theta(t)\sqrt{\Gamma_0}\exp\left[-\frac{1}{2\hbar}\int_{-\infty}^{t}\Gamma_0\Theta(t)dt\right] = \Theta(t)\sqrt{\Gamma_0}\exp\left[-\frac{\Gamma_0 t}{2\hbar}\right] \tag{63}$$

and putting it to the Wigner function (54) we obtain:

$$\begin{aligned} W_b(t,\epsilon_k) &= \frac{\Gamma_0}{2\pi\hbar^2}\int\Theta(t-\xi_2/2)\exp\left[-\frac{\Gamma_0}{2\hbar}(t-\xi_2)\right]\Theta(t+\xi_2/2)\exp\left[-\frac{\Gamma_0}{2\hbar}(t+\xi_2)\right]e^{-i\epsilon_k\xi_2/\hbar}d\xi_2 \\ &= \frac{\Gamma_0}{2\pi\hbar}e^{-\Gamma_0 t/\hbar}\Theta(t)\int_{-2t}^{+2t}e^{-i\epsilon_k\xi_2/\hbar}d\xi_2 = \frac{1}{h}\frac{2\Theta(t)e^{-\Gamma_0 t/\hbar}\sin(2t\epsilon_k/\hbar)}{\epsilon_k/\Gamma_0} \end{aligned} \tag{64}$$

It is apparent that we can slice out dimensionless part of Wigner function $\bar{W}_b$ as:

$$W_b(t,\epsilon_k) = \frac{1}{h}\bar{W}_b(\Gamma_0 t/\hbar, (\epsilon_k-\epsilon_d)/\Gamma_0) \tag{65}$$

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[5] We can observe it either in animation or in (50)
[6] It can be looked as generelasation of [7]

which from (64) is expressed as:

$$\bar{W}_b(t,\epsilon_k) = \frac{2}{\epsilon_k}\Theta(t)e^{-t}\sin(2\epsilon_k t) \tag{66}$$

The normalization to non-dimensionless variables t, ϵ_k also holds as $\int \bar{W}_b(t,\epsilon_k)dtd\epsilon_k = 1$ making it easily readable as quasi-probability distribution (For now old notation for all plots holds which has for 2π larger non-dimensional Wigner function values). By using (56) and (57) the Wigner function above is shown in Figure 2 with corresponding time and energy probability distributions.

B. Linear rise of tunneling barrier

Another case of interest is when the tunneling barrier is raised linearly [7] giving us emission protocol:

$$\Gamma(t) = \frac{\hbar}{\tau}e^{(t-t_e)/\tau} \tag{67}$$

$$\epsilon_d(t) = \epsilon_d(t_e) + \frac{\Delta_{ptb}}{\tau}(t - t_e) \tag{68}$$

Choosing the frames of reference for time and energy where $t_e = 0$ and $\epsilon_d(t_e) = 0$ we can consider simplified emission protocol as:

$$\Gamma(t) = \frac{\hbar}{\tau}e^{t/\tau} \tag{69}$$

$$\epsilon_d(t) = \frac{\Delta_{ptb}}{\tau}t \tag{70}$$

As in last section we start with expressing $B(t)$ from (13):

$$B(t) = \sqrt{\frac{\hbar}{\tau}}\exp\left[\frac{t}{2\tau} - i\frac{\Delta_{ptb}}{\tau\hbar}t^2 - \frac{1}{2}e^{t/\tau}\right] \tag{71}$$

which we put in the Wigner function (55) obtaining:

$$\begin{aligned}W_b(t,\epsilon_k) = \frac{1}{2\pi\hbar^2}\frac{\hbar}{\tau}\int \exp&\left[\frac{t-\xi_2/2}{2\tau} + i\frac{\Delta_{ptb}}{\tau\hbar}(t-\xi_2/2)^2 - \frac{1}{2}e^{(t-\xi_2/2)/\tau}\right]\\ \times\exp&\left[\frac{t+\xi_2/2}{2\tau} + i\frac{\Delta_{ptb}}{\tau\hbar}(t+\xi_2/2)^2 - \frac{1}{2}e^{(t+\xi_2/2)/\tau}\right]e^{-i\xi_2\epsilon_k/\hbar}d\xi_2\\ &= \frac{1}{2\pi\hbar\tau}\int \exp\left[\frac{t}{\tau} + i\frac{\Delta_{ptb}}{\hbar\tau}t\xi_2 - e^{t/\tau}\cosh(\xi_2/2\tau)\right]e^{-i\xi_2\epsilon_k/\hbar}d\xi_2\end{aligned} \tag{72}$$

We are going to consider dimensionless part of Wigner function $\bar{W}_b(t,\epsilon_k)$

$$W_b(t,\epsilon_k) = \frac{1}{h}\bar{W}_b\left(\frac{t-t_e}{\tau}, \frac{\tau}{\hbar}(\epsilon_k - \epsilon_d(t_e))\right) \tag{73}$$

for which also normalization remains. Expressing it from (72) one obtains:

$$\bar{W}_b(t,\epsilon_k) = \int \exp\left[t + i\frac{\Delta_{ptb}\tau}{\hbar}t\xi_2 - e^t\cosh(\xi_2/2)\right]e^{-i\xi_2\epsilon_k}d\xi_2 \tag{74}$$

Computed with adaptive Gaussian quadrature it is shown with time and energy probability distributions in Figure 3, where the effect of Δ_{ptb} have been analyzed.

C. Linear rise with harmonical modulation of tunneling barrier

And finally we consider the case with harmonical modulation of tunneling barriers when energy at the quantum dot remains constant which recently have became possible experimentally [2]. For this case emission protocol is given:

$$\epsilon_d(t) = \epsilon_d(t_e) \tag{75}$$

$$\Gamma(t) = \frac{\hbar}{\tau}\exp\left[\frac{t-t_e}{\tau} + A_\Gamma\sin\left(2\pi\frac{t-t_e}{\tau_0} + \phi_0\right)\right] \tag{76}$$

Choosing the frames of reference for time and energy where $\epsilon_d(t_e) = 0$ and $t_e = 0$ we obtain simplified emission protocol as:

$$\epsilon_d(t) = 0 \tag{77}$$

$$\Gamma(t) = \frac{\hbar}{\tau} \exp\left[\frac{t}{\tau} + A_\Gamma \sin\left(2\pi\frac{t}{\tau_0} + \phi_0\right)\right] \tag{78}$$

Since this problem is more complicated as previous ones some integrals can't be evaluated analytically. One of them is $X(t)$:

$$X(t) = \frac{1}{\hbar}\int_{-\infty}^{t} \Gamma(\tilde{t})d\tilde{t} = \frac{1}{\tau}\int_{-\infty}^{t} \exp\left[\frac{\tilde{t}}{\tau} + A_\Gamma \sin\left(2\pi\frac{\tilde{t}}{\tau_0} + \phi_0\right)\right] d\tilde{t} \tag{79}$$

which is evaluated numerically. Putting it and emission protocol in (13) we again express $B(t)$:

$$B(t) = \sqrt{\frac{\hbar}{\tau}} \exp\left[\frac{t}{2\tau} + \frac{A_\Gamma}{2}\sin(2\pi\frac{t}{\tau_0} + \phi_0) - \frac{1}{2}X(t)\right] \tag{80}$$

By putting it in (54) we obtain Wigner function:

$$\begin{aligned} W_b(t,\epsilon_k) = &\int e^{-i\epsilon_k T/\hbar} \exp\left[\frac{t - T/2}{2\tau} + \frac{A_\Gamma}{2}\sin\left(2\pi\frac{t - T/2}{\tau_0} + \phi_0\right) - \frac{1}{2}X(t - T/2)\right] \\ &\times \exp\left[\frac{t + T/2}{2\tau} + \frac{A_\Gamma}{2}\sin\left(2\pi\frac{t + T/2}{\tau_0} + \phi_0\right) - \frac{1}{2}X(t + T/2)\right] dT \\ = &\frac{1}{\tau h}\int e^{-i\epsilon_k T/\hbar} \exp\left[\frac{t}{\tau} + A_\Gamma \sin(2\pi t/\tau_0 + \phi_0)\cos(\pi T/\tau_0) - \frac{1}{2}\left(X(t - T/2) + X(t + T/2)\right)\right] dT \end{aligned} \tag{81}$$

We are going to consider dimensionless part of Wigner function and $X(t)$ as follows:

$$W_b(t,\epsilon_k) = \frac{1}{h}\bar{W}_b((t - t_e)/\tau, (\epsilon_k - \epsilon_d(t_e))\tau/\hbar) \tag{82}$$

$$X(t) = \bar{X}(t/\tau) \tag{83}$$

By using (79) and (81) one expresses dimensionless part of Wigner function:

$$\bar{X}(t) = \int_{-\infty}^{t} \exp\left[\tilde{t} + A_\Gamma \sin(2\pi\frac{\tau}{\tau_0}\tilde{t} + \phi_0)\right] d\tilde{t} \tag{84}$$

$$\bar{W}_b(t,\epsilon_k) = \int e^{-i\epsilon_k T} \exp\left[t + A_\Gamma \sin(2\pi\frac{\tau}{\tau_0}t + \phi_0)\cos(\pi T\tau/\tau_0) - \frac{1}{2}\left(\bar{X}(t - T/2) + \bar{X}(t + T/2)\right)\right] dT \tag{85}$$

where $\bar{X}(t)$ is evaluated separately. By observing localization of integrated function for T I evaluated it with trapezoidal rule with spacing much smaller than period of harmonic modulation $2\tau_0/\tau$. The computed Wigner function with its time and energy distributions are shown in Figure 4 where qualitative analysis of modulation parameters have been made.

D. Analysis of energy spectrum

Electron according to (56) is localized in time allowing us to estimate energy probability distribution numerically from (57). Also time probability distribution has been obtained numerically from left side of (56) and compared with result at the right shown in Figure 7 making us more confident about numerical convergence of the drawn Wigner function and as follows about energy probability distribution.

Firstly I was considering the effect of energy changes at the time of emission for which corresponding Wigner function can be seen in Figure 3 but energy distributions at the left of Figure 5. One sees that as quantum dot is raised faster electron in the lead becomes more spread. Also skewness of distribution becomes present. To analyse spreadness of electron due to Δ_{ptb} for each Wigner function the dispersion for both time and energy spectrum was calculated in usual fashion:

$$\Delta t = \sqrt{\langle (t - \langle t \rangle)^2 \rangle} \tag{86}$$

$$\Delta \epsilon_k = \sqrt{\langle (\epsilon_k - \langle \epsilon_k \rangle)^2 \rangle} \tag{87}$$

and then product was plotted shown in Figure 6. One observes it is always larger than theoretical minimum $\hbar/2$ due to uncertainty relations. Also we observe parabolic rise for small Δ_{ptb} which becomes linear as Δ_{ptb} becomes large.

After analyzing influence of Δ_{ptb} I considered the effect of harmonical modulation of tunneling barriers in a way that energy of quantum dot remains constant. First thing one notices in Figure 4 is the periodic structure both in energy and time axis directions. Here we will only consider energy probability distribution since other one with kinetic equation (3) is is already analyzed in great detail [8].

Harmonic modulation (76) introduces two free parameters amplitude A_Γ and its frequency $2\pi/\tau_0$ which were analyzed by setting one to constant shown in Figure 5. For constant frequency increase for amplitude results in production of Floquat sidebands and reduction for central maximum. Making at this point amplitude constant we see that the change of modulation frequency directly corresponds to the spread of the sidebands.

E. Notes on numerical approximations

The first case was solved analytically and therefore no numerical approximations was made. In the second case however numerical integration with adaptive Gaussian method had been used where as convergence condition was used estimated relative error about 0.1%. The integrator also was checked on the first case where no differences (visual) was observed.

In the third case however difficulties with oscillatory integrals rose. At the first try again the adaptive Gaussian method was used however it took computation time about hours for grid of size 100×100 to reach convergence making me to use other ways to numerically approximate integration. At first I observed that integrated function is localised which becomes more apparent if we rewrite Wigner function (55) as:

$$p_t(t) = \Gamma(t) e^{-\frac{1}{\hbar}\int_{-\infty}^{t} \Gamma(\tilde{t}) d\tilde{t}} \tag{88}$$

$$W_b(t, \epsilon_k) = \frac{1}{h} \int \sqrt{p_t(t - T/2) p_t(t + T/2)} \exp\left[-\frac{i}{\hbar} \int_{t-T/2}^{t+T/2} \epsilon_d(\tilde{t}) d\tilde{t} - \frac{i}{\hbar} T \epsilon_k \right] dT \tag{89}$$

where $p_t(t)$ is time probability density distribution. The $p_t(t)$ from one hand is normalized to unity $\int p_t(t) dt = 1$ but from other its values are $p_t(t) \geq 0$ from which comes it must also be localized[7] Let's consider ideal localisation such that $p_t(t)$ has nonzero values only in region $-a/2 < t < a/2$. This would allow to obtainWigner function (without approximations) by integration in finite region:

$$W_b(t, \epsilon_k) = \frac{1}{h} \int_{-a}^{+a} \sqrt{p_t(t - T/2) p_t(t + T/2)} \exp\left[-\frac{i}{\hbar} \int_{t-T/2}^{t+T/2} \epsilon_d(\tilde{t}) d\tilde{t} - \frac{i}{\hbar} T \epsilon_k \right] dT \tag{90}$$

In practical cases we approximate $p_t(t)$ as being localised in region $-a/2 < t < +a/2$ ifit integrated in this region as $P_{Norm} = \int_{-a/2}^{+a/2} p_t(\tilde{t}) d\tilde{t}$ gives normalisation close to 1. One can show that upper bound of error introduced due to localisation can be given:

$$\Delta W_b(t, \epsilon_k) \leq \frac{2}{h}(1 - P_{Norm}) \tag{91}$$

For evaluating the integral for finite region I used trapezodial rule where step size ΔT was choosen to take in acount the harmonic modulation period τ_0/τ. Nummerically both effects of approximation are shown in Figure 7.

VIII. PLOTS

[7] I can't imagine counter example. Nevertheless $p_t(t)$ analytical expression for harmonical modulation was clearly localized.

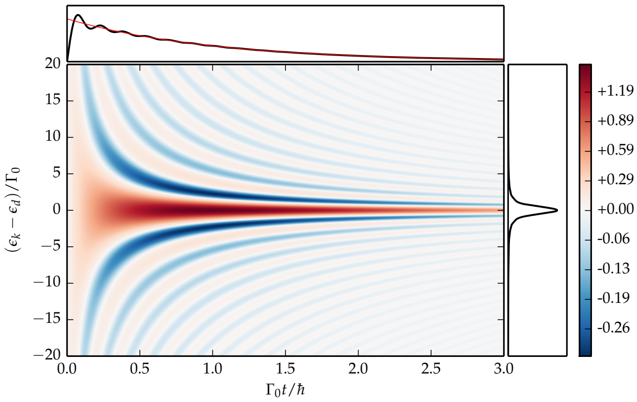

FIG. 2: Wigner function for instantaneous rise of tunneling barrier (66). The time and energy probability distributions obtained with assumption of localization in region drawn is shown with black solid lines. The comparison has been made for time probability distribution which is obtained analytically with (56) shown with red line, where one sees the effect of not using region large enough of drawing it in which the normalization is $\int_{region} W_b(t, \epsilon_k) dt d\epsilon_k = 0.93$. Further all numerical calculations for time and energy probability distributions are done with normalization at least 0.99.

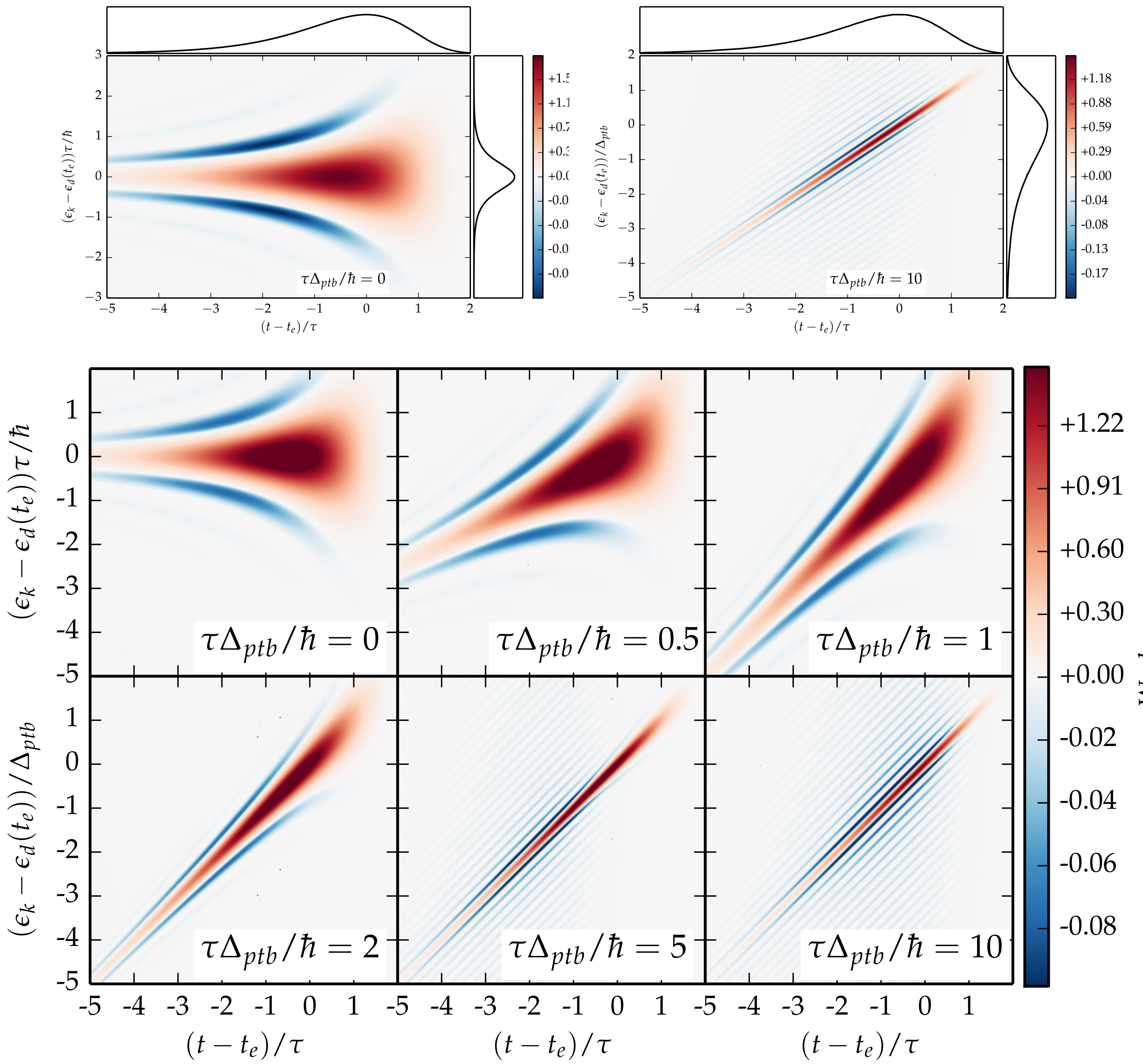

FIG. 3: Wigner function for linear rise of tunneling barriers (72). At the top left is shown wave packet when energy of quantum dot does not changes. At the right the limit $\Delta_{ptb} \to +\infty$ with changed energy scale to Δ_{ptb} is drawn. It shows that energy distribution becomes the reflection of time probability distribution since electron's energy dispersion becomes much smaller than energy changes introduced with Δ_{ptb}. Since both of these seemed quite different then I have drawn them as progression of Δ_{ptb} at the bottom.

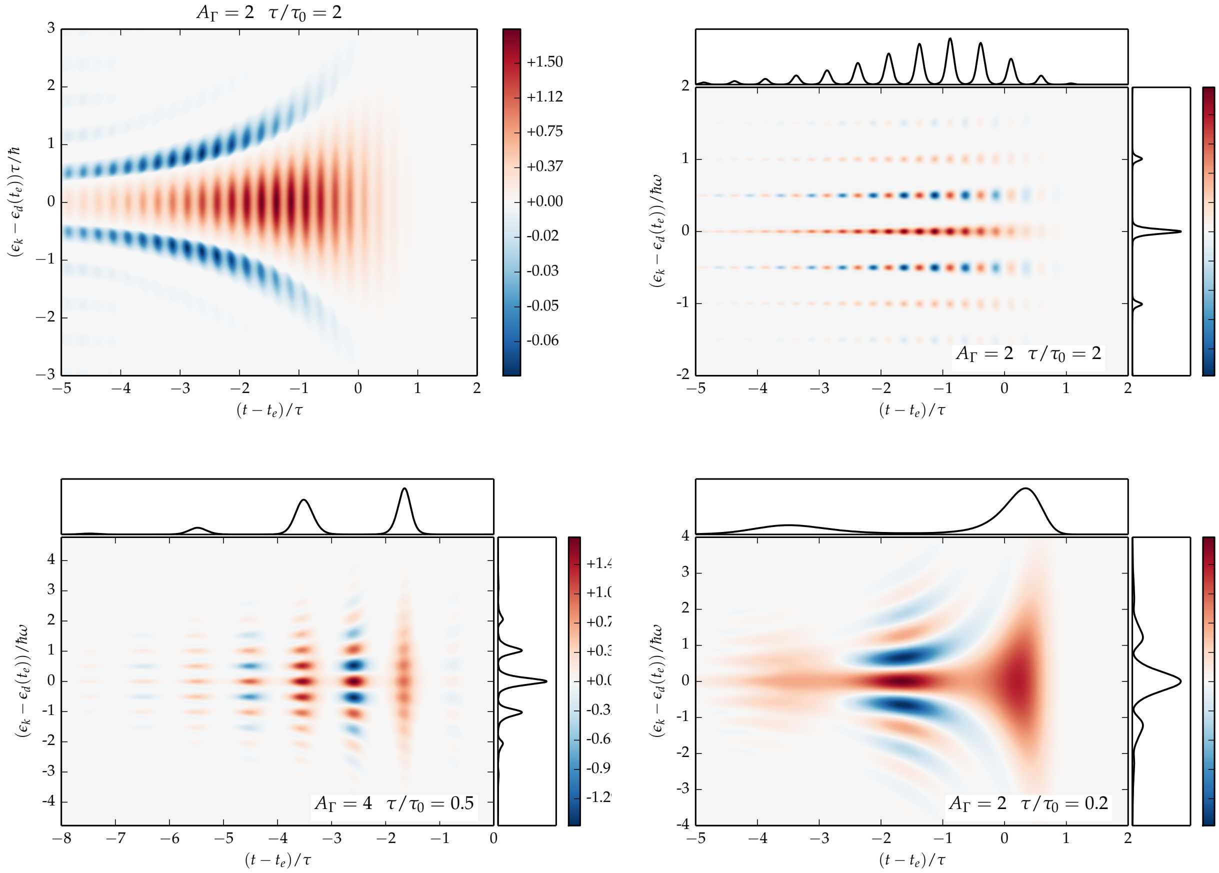

FIG. 4: Wigner function for harmonic rise of tunneling barriers (85). At the top left I have shown the correspondence with case where energy of quantum dot remains constant considered in Figure 3. Setting modulation phase $\phi_0 = 0$ the remaining two modulation parameters amplitude A_Γ and frequency τ/τ_0 was changed. It can be seen that there are periodic structure in both time and energy axes directions where the latter one is called Floquet sidebands.

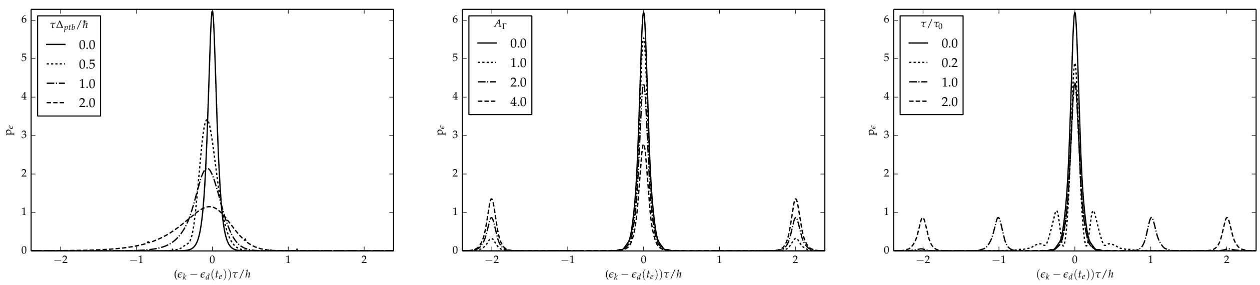

FIG. 5: Energy probability distributions for different emission protocol parameters. First one is considered without harmonic modulation or $A_\Gamma = 0$, second one with steady energy of quantum dot or Δ_{ptb} and $\tau/\tau_0 = 2$, where the third one $\Delta_{ptb} = 0$, $A_\Gamma = 2$

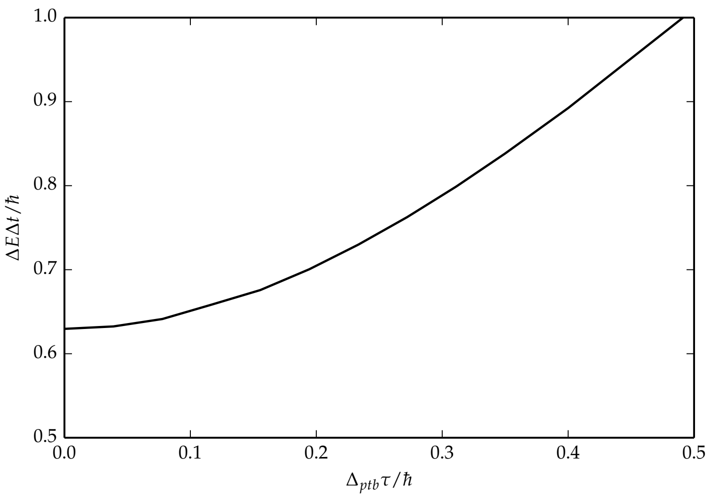

FIG. 6: The spread of wave packet as function of speed at which energy in quantum dot is raised.

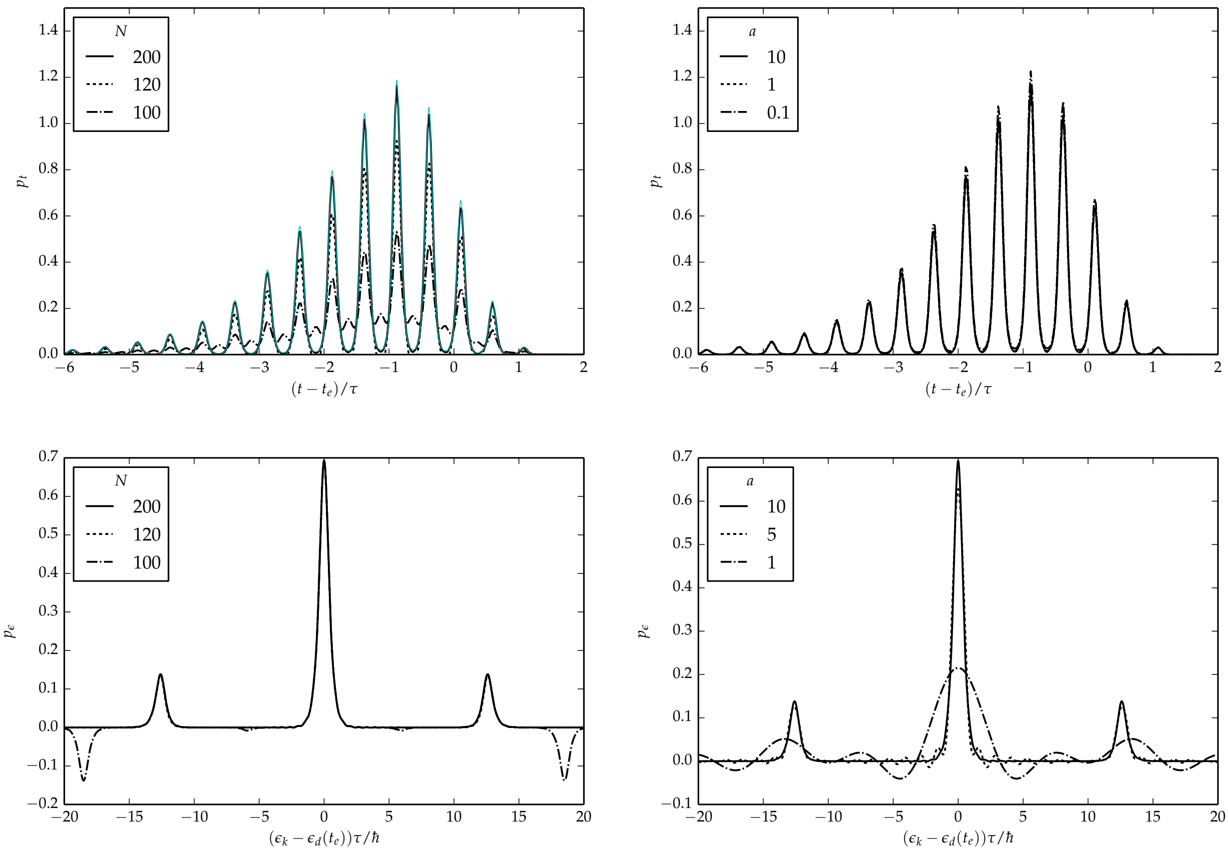

FIG. 7: At the top we see direct comparison with time probability distribution obtained from Wigner function with parameters $A_\Gamma = 2$, $\tau/\tau_0 = 2$ numerically by applying property (56) and also analytically from kinetic equation (3) which was taken from [8]. It shows fast convergence as spacing of T becomes smaller than $2\tau_0/\tau$ at numerical evaluation of integral (85) with trapezoidal rule. The asumption of localisation for integration variable T however does not show signs in time probability distribution (as expected), while in energy probability distribution we see that localisation parameter a must be choosen with respect to normalization $P_{Norm} = \int_{-a/2}^{+a/2} p_t(t)dt$.

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